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THESIS

THE APPLICATION OF THE KALMAN FILTER
TO THE
SONOBUOY REFERENCE SYSTEM ON THE S3A

by

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ABSTRACT

In 1960 R. E. Kalman introduced a least squares concept that gives optimal estimates for the state of some dynamic systems. Included is a brief historical introduction leading to his work, a summary of his work, and the application of the theory to the sonobuoy reference system used on the S3A aircraft. Also, a tutorial development of certain quantities used in the filter is presented in appendixes A and B.

TABLE OF CONTENTS

I. INTRODUCTION

Aircraft launched sonobuoys form an important and growing role in ASW; however, it is necessary to know the position of the sonobuoys to fully utilize the information they transmit. In the past, accurate positions were available only if the aircraft flew over the top of the buoy at low altitudes. For a large buoy pattern such a method consumes time which could be used for other aspects of the mission. The Navy's newest ASW aircraft, the S3A, will use a unique technique called the Sonobuoy Reference System (SRS) to estimate the sonobuoy's location. The SRS uses an interferometry principle coupled with a modern digital computer to update buoy positions. To increase the accuracy of the estimated position, the data from the SRS are processed through a Kalman filter.

The Kalman filter is a recent advance in estimation theory that has been widely used in the aerospace industry. The term, filter, was advanced by electrical engineers to describe an input-output system. The Kalman filter was designed as a linear filter. A linear filter is a process which converts an input to an output in such a way that the output resulting from two simultaneous inputs is the addition of the two corresponding outputs [1].

Historically, estimation theory has been concerned with estimating the values of system parameters based on some observation of the system. The first major estimation

technique was developed independently by A. M. Legendre and K. F. Gauss. Although Legendre published his results in 1806 while Gauss published his results in 1809, Gauss is generally credited with inventing the method of least squares because he derived the method from fundamental principles and claimed to have used the technique to determine the parameters describing the orbital motion of celestial bodies in 1795. Gauss' parameters were the most probable value of the unknown quantities based on the data available. His definition of most probable values was;

> The most probable value of the desired parameters will be that in which the sum of squares of the differences between the actually observed and computed values multiplied by numbers that measure the degree of precision is a minimum [2].

Following Gauss' book there was a long period where least squares was the only widely accepted method for estimation.

In 1912 R. A. Fisher published the method of maximum likelihood and thereby established the present theoretical background for estimation theory [3]. Maximum likelihood estimation is cast in probability theory which requires information on the probability distribution describing the object being measured. For an important class of problems, there is a direct correspondence between the maximum likelihood and the least squares methods. Assuming the object being observed can be described by a gaussian distribution, the two theories give identical results for the respective estimators.

Following Fisher's work there was considerable effort in the field of communications to eliminate random noise from the signal. A. N. Kolmogorov and N. Wiener, in 1941 and 1942, independently developed a linear minimum mean-square theory that provided the best separation of signal and noise [4,5]. The Wiener-Kolmogorov theory is a least squares process which differs from Gauss' theory primarily in that it allows the signal to vary with time with known statistical qualities and the filter is required to be linear. The main problem associated with their filter is the difficulty in calculating the required parameters especially for a complex system. Their results formed the basis for the subsequent work of R. E. Kalman.

II. KALMAN THEORY

Kalman's first paper on his filter theory was published in 1960 [6]. A summary of his results follows in a discrete time version which is the form used by the SRS. Notation used in this paper and appendixes is; capital letters and small letters represent vectors or matrices and their components, respectively, subscripts refer to the time period being considered, carets indicate values that have been extrapolated through time and bars indicate that the values have been smoothed or weighted by some measurement information. The equations, describing the system being observed, must be of the form

$$X_i = \Phi_{i,i-1} X_{i-1} + U_i. \tag{1}$$

X_i is a nx1 vector that describes the state of the system at time i. Each component of X_i is a state variable; therefore, X_i is called the state vector. $\Phi_{i,i-1}$ is a nxn transition matrix that determines how the system propagates from time i-1 to i. U_i is a nx1 vector that is a gaussian random process which represents the uncertainties of the state equations in describing the true state. The statistical properties of U_i are

$$E[U_i] = 0 \qquad \text{for all } i, \tag{2}$$

$$E[U_i U_j^T] = 0 \qquad \text{for } i \neq j, \tag{3}$$

$$E[U_i U_i^T] = V_{U_i} \qquad \text{for all } i. \tag{4}$$

System observations are given by

$$Z_i = M_i X_i + W_i \tag{5}$$

Z_i is a pxl ($p \leq n$) observation vector. Each component of Z_i is an observation on the system at time i that is related to the state variables through M_i, a pxn measurement matrix. Kalman theory requires M_i to be such that the observations are linear combinations of the state variables. W_i is a pxl vector that is a gaussian random process which represents the uncertainties of the measuring processes at time i. The statistical properties of W_i are

$$E[W_i] = 0 \qquad \text{for all } i, \tag{6}$$

$$E[W_i W_j^T] = 0 \qquad \text{for } i \neq j, \tag{7}$$

$$E[W_i W_i^T] = V_{W_i} \qquad \text{for all } i. \tag{8}$$

In addition, U and W are assumed independent therefore,

$$E[W_i U_j^T] = 0 \qquad \text{for all } i,j. \tag{9}$$

Kalman proposed to estimate the state X_i from the last estimate $\overline{X}_{i-1}$ and the observation Z_i in such a way as to minimize the mean-square error in the estimate. That is,

$$E[(X_i - \overline{X}_i)^2] \tag{10}$$

is to be minimized. The solution to this problem has become known as the Kalman filter. The optimal estimate of the state at time i is

$$\overline{X}_i = \hat{X}_i + K_i[Z_i - M_i\hat{X}_i] \tag{11}$$

where $$\hat{X}_i = \Phi_{i,i-1}\overline{X}_{i-1} \; . \tag{12}$$

The gain matrix K_i is determined by

$$K_i = \hat{C}_i M_i^T [M_i \hat{C}_i M_i^T + V_{W_i}]^{-1} \tag{13}$$

where $$\hat{C}_i = \Phi_{i,i-1}\overline{C}_{i,i-1}\Phi^T_{i,i-1} + V_{U_{i-1}} \tag{14}$$

and $$\overline{C}_{i-1} = \hat{C}_{i-1} - K_{i-1}M_{i-1}\hat{C}_{i-1} \tag{15}$$

where C_i is the nxn covariance matrix of the state variables. In words, the optimal estimate of X_i is the optimal estimate at time i-1 propagated to time i and then modified by the weighting matrix K_i times the difference between the actual observation at time i and the estimated value of the observation at time i. Equations (11) through (15) are the basic equations of the Kalman filter. Initial requirements for the filter are X_o, C_o, and the variances of U and W.

Although the Kalman filter was designed for linear systems it can be modified for use with a nonlinear system as in the case of the SRS. Typically, the state equations and the observations are nonlinear and both must be "linearized" by making linear approximations of the equations to use in the filter. Such a method is referred to as the extended Kalman filter. The extended Kalman filter equations are analogous to the linear model equations [8]. Since the

SRS utilizes linear state equations, the equations are extended to include only nonlinear observations. The state equation is

$$X_i = \Phi_{i,i-1} X_{i-1} \tag{16}$$

The gaussian process U_{i-1} was omitted from eq. (16) to facilitate later development. The omission is equivalent to assuming that the actual system has been modeled error free. The observation is

$$Z_i = h(X_i) + W_i \tag{17}$$

where h(·) is a nonlinear function of the state variables. The estimate of the state at time i is

$$\overline{X}_i = \hat{X}_i + K_i[Z_i - h(\hat{X}_i)] \tag{18}$$

The gain matrix is

$$K_i = \hat{C}_i D_i^T [D_i \hat{C}_i D_i^T + V_{W_i}]^{-1} \tag{19}$$

where D_i is a nxp matrix of the partial derivatives of h(·), with respect to the state variables, evaluated at $X_i = \hat{X}_i$ also,

$$\hat{C}_i = \Phi_{i,i-1} \overline{C}_{i-1} \Phi^T_{i,i-1} \tag{20}$$

$$\overline{C}_{i-1} = \hat{C}_{i-1} - K_{i-1} D_{i-1} \hat{C}_{i-1} \quad . \tag{21}$$

The approximations used in the extended Kalman filter can lead to unsatisfactory solutions due to divergence. Divergence occurs when C_k, the covariance matrix of the state variables, approaches zero. This results in the weighting factors becoming unjustifiably small causing the incoming data to be weighted too little [7]. Thus, the solution becomes dominated by the time propagation of the state equations. One source of divergence is system equations that do not properly model the physical system. This situation can occur if the linear approximations implicit in eq. (19) are not adequate in describing the nonlinear system. These approximations are analogous to discarding all the second degree and higher terms in a Taylor's series expansion of the nonlinear equations. Several approaches to shaping the state covariance matrix to avoid filter divergence are discussed in [7].

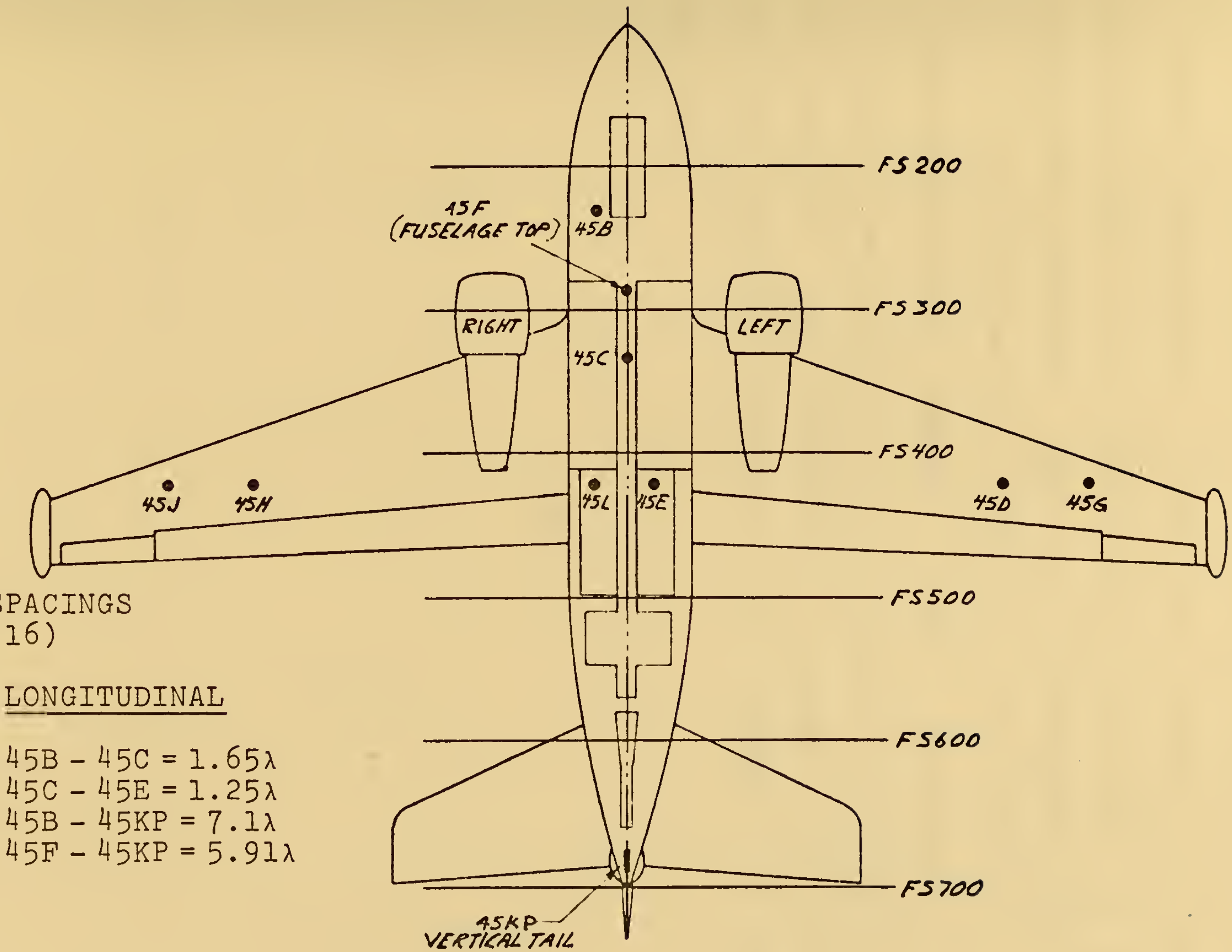
III. SONOBUOY REFERENCE SYSTEM

The SRS was designed to periodically update the position of the sonobuoys based on information received over an antenna system. Figure 1 shows the location of the antennas. These antennas provide three possible observations on the sonobuoy.

1. Longitudinal direction cosine.
2. Transverse direction cosine.
3. Range from aircraft to buoy.

Range information is available in limited conditions. The normal mode of operation is with azimuth information. The azimuth measurements are calculated from the phase difference of the sonobuoy radio signals measured between the ends of antenna pairs. The antennas, with the shortest spacings, are used to give the general relative bearing to the buoy. Once this has been determined the computer selects a pair of antennas that are further apart and refines the bearing. This is a repetitive process which continues until the antenna pair is separated as far as possible. Antennas are located on the top of the fuselage and in the vertical tail to allow buoy signals to be received while the aircraft is banked in a turn.

During normal operation, the SRS sequentially checks each of the 16 VHF channels to determine if any signals are being received. If no signals are being received on a



ANTENNA SPACINGS
(CHANNEL 16)

TRANSVERSE	LONGITUDINAL
45E - 45L = .4λ	45B - 45C = 1.65λ
45G - 45D = 1.21λ	45C - 45E = 1.25λ
45H - 45J = 1.21λ	45B - 45KP = 7.1λ
45D - 45E = 3.69λ	45F - 45KP = 5.91λ
45L - 45H = 3.69λ	
45G - 45J = 10λ	

FIGURE 1
(Lockheed Furnished)

channel that has a buoy assigned to it, the buoy location and the uncertainty of its location is updated by propagating the last estimate. Should the aircraft be within receiving range of two or more buoys on the same VHF channel, no data are processed. When the SRS determines that a signal can be processed for information, the direction cosine is calculated and a command is sent to the aircraft navigation system to determine the orientation and position of the aircraft. This information is passed to the filter, processed and then displayed to the crewmen on a cathode ray tube.

IV. KALMAN FILTER FOR THE SRS

The mathematical formulation of the problem utilizes two coordinate systems, the local and the aircraft system. The local coordinate system is a three-dimensional, right-handed system with the x-axis pointing eastward, the X-Y plane target to the surface of the earth and the origin arbitrarily fixed. The aircraft attitude coordinate system is used to reference the aircraft orientation in roll, pitch and heading. The aircraft attitude can be transformed into local coordinate values through a rotation matrix.

The state equations for the system are based on the local coordinate system. The model of the buoy motion used in the SRS is

$$x_{i+1} = x_i + \dot{x}_i \, dt \tag{22}$$

$$y_{i+1} = y_i + \dot{y}_i \, dt \tag{23}$$

$$\dot{x}_{i+1} = \dot{x}_i \tag{24}$$

$$\dot{y}_{i+1} = \dot{y}_i \tag{25}$$

where dt is the time lapse between observations. In matrix notation they can be expressed as

$$X_{i+1} = \Phi X_i \tag{26}$$

where

$$X_i = \begin{pmatrix} x_i \\ y_i \\ \dot{x}_i \\ \dot{y}_i \end{pmatrix} \tag{27}$$

and

$$\Phi = \begin{pmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{28}$$

The subscript on the transition matrix, Φ, has been suppressed since it is not changing with time. Also, the buoy motion as indicated by eqs. (22) thru (25) lies in a plane parallel to the X-Y plane of the local coordinate system. This does not imply that the curvature of the earth is not taken into account only that buoy motion in the Z-direction is negligible.

The observation, Z, would be a 3x1 vector assuming the three possible observations were available simultaneously. This would require excessive computer time to invert the resulting 3x3 matrix in eq. (19) for calculating the gain matrix. The SRS uses a sequential updating method to avoid this problem. Whenever multiple observations are available, they are processed independently thereby reducing W and Z to scalars. The observation for the direction cosine input is

$$Z_i = \cos \theta_i + W_i \tag{29}$$

where W_i is assumed to meet the requirements of the gaussian noise in eq. (5). The observation is related to the state variables through the geometry in figure 2.

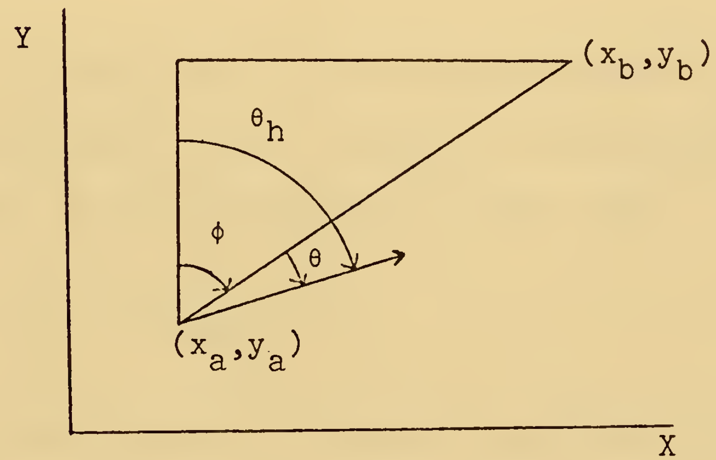


Figure 2

θ_h is the aircraft heading, (x_a, y_a) are the coordinates of the aircraft position and (x_b, y_b) are the coordinates of the sonobuoy. Cos θ can be expressed as

$$\cos\theta = \cos(\theta_h - \phi) \tag{30}$$

and since

$$\cos\phi = \frac{y_b - y_a}{R}, \tag{31}$$

$$\sin\phi = \frac{x_b - x_a}{R} \tag{32}$$

where $$R = [(x_b - x_a)^2 + (y_b - y_a)^2]^{1/2} \tag{33}$$

it follows that

$$\cos\theta_i = \frac{y_{bi}-y_{ai}}{R_i}\cos\theta_{hi} + \frac{x_{bi}-x_{ai}}{R_i}\sin\theta_{hi}. \quad (34)$$

Since R_i is nonlinear in the state variables x_b and y_b and considering θ_{hi}, x_{ai}, and y_{ai} as known quantities, the observation can be expressed as in eq. (17)

$$Z_i = h(x_i,y_i) + W_i \quad (35)$$

where x_i and y_i refer to the buoy coordinates at time i.

The Kalman filter equations are given by equations (16) thru (21) with $\phi_k = \phi$ for all k. The state variables covariance matrix would be

$$C = \begin{pmatrix} c(1) & c(2) & c(3) & c(4) \\ c(2) & c(5) & c(6) & c(7) \\ c(3) & c(6) & c(8) & c(9) \\ c(4) & c(7) & c(9) & c(10) \end{pmatrix} \quad (36)$$

where $c(1) = \mathrm{Var}\,[x]$ (37)

$c(5) = \mathrm{Var}\,[y]$ (38)

$c(8) = \mathrm{Var}\,[\dot{x}]$ (39)

$c(10) = \mathrm{Var}\,[\dot{y}]$ (40)

and the remaining elements are the respective covariances, e.g.

$$c(6) = \mathrm{Cov}\,[y,\dot{x}]. \quad (41)$$

However, since the modeled system has no uncertainty built into it some method must be used to prevent divergence. To insure that C would not become too small c(8) and c(10) are replaced by Var [x] + $(dAdt)^2$ and Var [y] + $(dAdt)^2$ prior to the propagation of the C matrix in time. The value of dA is parameterized at .002 ft/sec^2 and it represents an acceleration uncertainty [9]. Examination of eq. (34) reveals that the partial derivatives with respect to the velocity state vectors will be zero. Accordingly,

$$D = (h_x \;\; h_y \;\; 0 \;\;\; 0) \tag{42}$$

h_x and h_y are derived in Appendix B.

The required initial values for aircraft launched buoys are:

1. State Vector. Coordinates are the aircraft drop coordinates and the component velocities are zero unless additional information is known.
2. Covariance. Off diagonal elements are zero. c(1) and c(5) are H^2 where H is the altitude of the aircraft in nautical miles. c(8) and c(10) are (5 knots)^2.
3. Measurement variance. This quantity depends upon which sector relative to the aircraft the buoy is located, W = $(1/3.6)^2$ except when the buoy is within 35° of either side of the rear longitudinal line when W = $(1/2.4)^2$.

V. SUITABILITY OF THE KALMAN FILTER FOR THE SRS

There are two requirements for a filter used in real-time data processing. First, the storage and computational demands on the computer must be attainable and second, the accuracy requirements of the problem must be met or exceeded. The computer demands of a Kalman filter are high when compared to other methods [10]; however, the SRS uses several techniques to reduce this demand. These techniques are directly related to the accuracy of the filter. Although the Kalman filter is optimal for a linear system, there is no guarantee that the real situation is modeled correctly. The errors introduced in modeling or in simplifying calculations may result in marginal or unsatisfactory filter performance. An important result of the Kalman filter for computer programming is that all the information from $Z_o, Z_1 \ldots\ldots\ldots Z_{i-1}$ is contained in $\dot{X}_{i-1}$ eliminating the need to store the old observations. In addition, the recursive nature of the equations make them readily adaptable for use in a computer. The basic steps in programming the filter are:

1. Initialize X, C, and W.
2. Extrapolate C (eq. 20).
3. Compute weighting coefficients (eq. 19).
4. Obtain measurement and estimate the state vector (eqs. 17 and 12).

5. Update C to reflect the last estimate (eq. 21).

6. Return to step 2.

Kalman's assumptions, of a linear system with additive gaussian noise, are approximated by the SRS. The nonlinear observation has been discussed previously. The gaussian distributed bearing error assumption may be significantly violated for certain flight conditions. Whenever the line of sight between the aircraft and sonobuoy does not change significantly as compared to the data rate, the bearing errors may become biased due to multipath effects. Multi-pathing is caused by the radio signal reflecting off the aircraft surfaces prior to reaching an antenna. This results in an apparent new position for the sonobuoy. Such errors would not be gaussian distributed and would not be compensated for by the filter. In an effort to reduce the biasing, a 1/7 scale model was used to empirically determine constants to apply to the observations. When the SRS determines what sector the sonobuoy lies in relative to the aircraft, the multipath constant for that sector is added to the observation to remove some of the biasing. Whether the constants from the scale model will be sufficient or whether constants for each aircraft will need to be determined will have to be decided after the SRS is flight tested in early 1973.

APPENDIX A

DERIVATION OF GAIN MATRIX COMPONENTS

The components of the gain matrix, K_i, are calculated directly from the principle of minimum mean square error. Scalar equations are used instead of matrix equations to facilitate gaining insight into the various relationships and to emphasize the approximations. Only one weighting constant is derived; however, the remaining constants can be obtained in a similar manner.

The equation, for the smoothed estimate of the x-coordinate of the buoy, is

$$\bar{x}_{i+1} = \hat{x}_{i+1} + k_{1i}[z_{i+1} - h(\hat{x}_{i+1},\hat{y}_{i+1})] \qquad \text{(A1)}$$

k_{1i} is to be calculated so that

$$E[(\bar{x}_{i+1} - x_{i+1})^2] \qquad \text{(A2)}$$

is a minimum.

Now

$$\begin{aligned} \bar{x}_{i+1} - x_{i+1} &= \hat{x}_{i+1} + k_{1i}[z_{i+1}-h(\hat{x}_{i+1},\hat{y}_{i+1})] - x_{i+1} \\ &= \bar{x}_i + \dot{\bar{x}}_i dt + k_{1i}[h(x_{i+1},y_{i+1})+w_{i+1} \\ &\quad -h(\bar{x}_i+\dot{\bar{x}}_i dt,\bar{y}_i+\dot{\bar{y}}_i dt)] - x_i - \dot{x}_i dt \end{aligned} \qquad \text{(A3)}$$

let

$$\varepsilon_{x_i} = \bar{x}_i - x_i \qquad \text{and} \qquad \dot{\varepsilon}_{x_i} = \dot{\bar{x}}_i - \dot{x}_i \qquad \text{(A4),(A5)}$$

substitution into eq. (A3) yields

$$\varepsilon_{x_{i+1}} = \varepsilon_{x_i} + \dot{\varepsilon}_{x_i} dt + k_{1i}[h(x_i+\dot{x}_i dt, y_i+\dot{y}_i dt)+w_{i+1}$$

$$-h(\bar{x}_i+\dot{\bar{x}}_i dt, \bar{y}_i+\dot{\bar{y}}_i dt)] , \tag{A6}$$

in addition,

$$h(x_i+\dot{x}_i dt, y_i+\dot{y}_i dt) = h(\bar{x}_i-\varepsilon_{x_i}+(\dot{\bar{x}}_i-\dot{\varepsilon}_{x_i})dt, \bar{y}_i-\varepsilon_{y_i}+(\dot{\bar{y}}_i-\dot{\varepsilon}_{y_i})dt)$$

$$= h(\bar{x}_i+\dot{\bar{x}}_i dt-\varepsilon_{x_i}-\dot{\varepsilon}_{x_i} dt, \bar{y}_i+\dot{\bar{y}}_i dt-\varepsilon_{y_i}-\dot{\varepsilon}_{y_i} dt) . \tag{A7}$$

Using Taylor series expansion about the i^{th} smooth estimate, the expression for $h(\cdot,\cdot)$ can be approximated by

$$h(\bar{x}_i+\dot{\bar{x}}dt, y_i+\dot{\bar{y}}dt) + h_x(\bar{x}_i+\dot{\bar{x}}_i dt, \bar{y}_i+\dot{\bar{y}}_i dt)(-\varepsilon_{x_i}-\dot{\varepsilon}_{x_i} dt)$$

$$+ h_y(\bar{x}_i+\dot{\bar{x}}dt, \bar{y}_i+\dot{\bar{y}}_i dt)(-\varepsilon_{y_i}-\dot{\varepsilon}_{y_i} dt) \tag{A8}$$

where

$$h_x(\cdot,\cdot) = \frac{\partial h(\cdot,\cdot)}{\partial x} \quad \text{and} \quad h_y(\cdot,\cdot) = \frac{\partial h(\cdot,\cdot)}{\partial y} \qquad \text{(A9),(A10)}$$

Since the arguments for $h(\cdot,\cdot)$ do not change and for ease of manipulation, the arguments of $h(\cdot,\cdot)$ are suppressed and equation (A8) becomes

$$h + h_x(-\varepsilon_{x_i}-\dot{\varepsilon}_{x_i} dt) + h_y(-\varepsilon_{y_i}-\dot{\varepsilon}_{y_i} dt) \tag{A11}$$

Substituting eq. (A11) into (A6) and squaring both sides yields

$$\varepsilon^2_{x_{i+1}} = \varepsilon^2_{x_i} + \dot{\varepsilon}^2_{x_i} dt^2 + k^2_{1i}[\quad]^2 + 2\varepsilon_{x_i}\dot{\varepsilon}_{x_i} dt + 2\varepsilon_{x_i} k_{1i}[\quad]$$

$$+ 2\dot{\varepsilon}^2_{x_i} dt\; k_{1i}[\quad] \tag{A12}$$

where

$$[\quad]^2 = w^2_{i+1} + (\varepsilon_{x_i} + \dot{\varepsilon}_{x_i} dt)^2 h^2_x + (\varepsilon_{y_i} + \dot{\varepsilon}_{y_i} dt)^2 h^2_y$$

$$- 2w_{i+1}(\varepsilon_{x_i} + \dot{\varepsilon}_{x_i} dt)h_x - 2w_{i+1}(\varepsilon_{y_i} + \dot{\varepsilon}_{y_i} dt)h_y$$

$$+ 2(\varepsilon_{x_i} + \dot{\varepsilon}_{x_i} dt)h_x (\varepsilon_{y_i} + \dot{\varepsilon}_{y_i} dt)h_y \;. \tag{A13}$$

Taking the expected value and defining

$$\sigma^2_{x_i} = E[\varepsilon^2_{x_i}] \qquad \sigma^2_{\dot{x}_i} = E[\dot{\varepsilon}^2_{x_i}]$$

$$\sigma^2_{x\dot{x}_i} = E[\varepsilon_{x_i}\dot{\varepsilon}_{x_i}] \qquad \sigma^2 = E[w^2_{i+1}] \qquad \text{(A14) thru (A20)}$$

$$E[w_{i+1}] = E[w_{i+1}\varepsilon_{x_i}] = E[w_{i+1}\dot{\varepsilon}_{x_i}] = 0$$

with similar expression for the y-quantities yields

$$\sigma^2_{x_{i+1}} = \sigma^2_{x_i} + \sigma^2_{\dot{x}_i} dt^2 + k^2_{1i}[\sigma^2 + (\sigma^2_{x_i} + 2\sigma^2_{x\dot{x}_i} dt + \sigma^2_{\dot{x}_i} dt^2)h_x^2$$

$$+ (\sigma^2_{y_i} + 2\sigma^2_{y\dot{y}_i} dt + \sigma^2_{\dot{y}_i} dt^2)h_y^2 + 2(\sigma^2_{xy_i} + \sigma^2_{\dot{x}y_i} dt + \sigma^2_{x\dot{y}_i} dt$$

$$+ \sigma^2_{\dot{x}\dot{y}_i} dt^2)h_x h_y] + 2\sigma^2_{x\dot{x}_i} dt - 2k_{1i}h_x(\sigma^2_{x_i} + \sigma^2_{x\dot{x}_i} dt)$$

$$- 2k_{1i}h_y(\sigma^2_{xy_i} + \sigma^2_{x\dot{y}_i} dt) - 2k_{1i}h_x dt(\sigma^2_{x\dot{x}_i} + \sigma^2_{\dot{x}_i} dt)$$

$$- 2k_{1i}h_y dt(\sigma^2_{\dot{x}y_i} + \sigma^2_{\dot{x}\dot{y}_i} dt). \qquad \text{(A21)}$$

Solving eq. (A21) for the k_{1i} which minimizes $\sigma^2_{x_{i+1}}$ using standard calculus techniques results in

$$k_{1i} = \frac{h_x[\sigma^2_{x_i} + 2\sigma^2_{x\dot{x}_i} dt + \sigma^2_{\dot{x}_i} dt^2] + h_y[\sigma^2_{xy_i} + \sigma^2_{x\dot{y}_i} dt + \sigma^2_{\dot{x}y_i} dt + \sigma^2_{\dot{x}\dot{y}_i} dt^2]}{B} \qquad \text{(A22)}$$

where

$$B = \sigma^2 + h_x^2(\sigma^2_{x_i} + 2\sigma^2_{x\dot{x}_i} dt + \sigma^2_{\dot{x}_i} dt^2) + 2h_x h_y(\sigma^2_{xy_i} + \sigma^2_{\dot{x}y_i} dt + \sigma^2_{x\dot{y}_i} dt$$

$$+ \sigma^2_{\dot{x}\dot{y}_i} dt^2) + h_y^2(\sigma^2_{y_i} + 2\sigma^2_{y\dot{y}_i} dt + \sigma^2_{\dot{y}_i} dt^2). \qquad \text{(A23)}$$

Similar results are obtained if the following matrices are substituted into eq. (19) and expanded:

$$C_i = \begin{pmatrix} 1 & 0 & dt & 0 \\ 0 & 1 & 0 & dt \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma^2_{x_i} & \sigma^2_{xy_i} & \sigma^2_{x\dot{x}_i} & \sigma^2_{x\dot{y}_i} \\ \sigma^2_{xy_i} & \sigma^2_{y_i} & \sigma^2_{\dot{x}y_i} & \sigma^2_{y\dot{y}_i} \\ \sigma^2_{x\dot{x}_i} & \sigma^2_{\dot{x}y_i} & \sigma^2_{\dot{x}_i} & \sigma^2_{\dot{x}\dot{y}_i} \\ \sigma^2_{x\dot{y}_i} & \sigma^2_{y\dot{y}_i} & \sigma^2_{\dot{x}\dot{y}_i} & \sigma^2_{\dot{y}_i} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ dt & 0 & 1 & 0 \\ 0 & dt & 0 & 1 \end{pmatrix} \tag{A24}$$

$$D_i = (h_x \;\; h_y \;\; 0 \;\; 0) \tag{A25}$$

$$W_i = \sigma^2 \quad . \tag{A26}$$

APPENDIX B

DERIVATION OF PARTIAL DERIVATIVES

Figure B1 illustrates the geometry for the derivation of the partial derivatives of the measurement, cos θ.

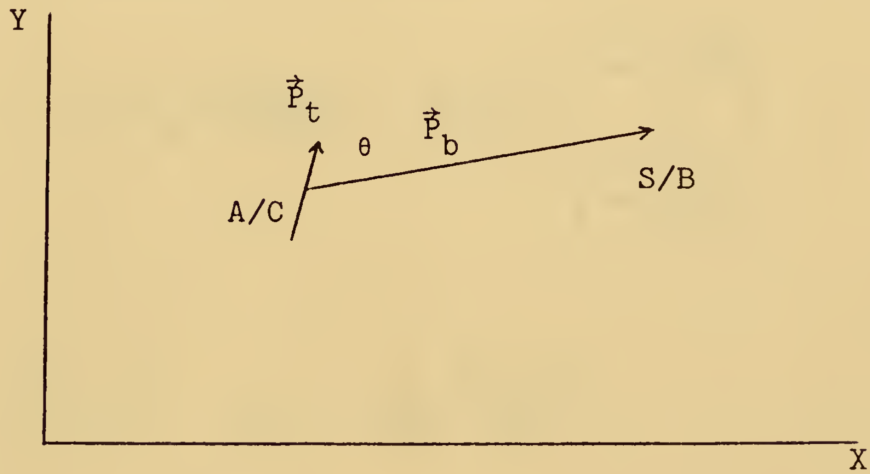


FIGURE B1

$\vec{P}_b$ is the vector from the aircraft to the sonobuoy.
$\vec{P}_t$ is the vector representing the antenna pair making the observation.

$$\vec{P}_b = (x_b - x_a)\tilde{i} + (y_b - y_a)\tilde{j} \tag{B1}$$

$$\vec{P}_t = (x_{t_1} - x_{t_2})\tilde{i} + (y_{t_1} - y_{t_2})\tilde{j} \tag{B2}$$

The corresponding unit vectors are

$$\tilde{P}_b = \frac{x_b - x_a}{R}\,\tilde{i} + \frac{y_b - y_a}{R}\,\tilde{j} \tag{B3}$$

$$\tilde{P}_t = \frac{x_{t_1}-x_{t_2}}{k_t}\,\tilde{i} + \frac{y_{t_1}-y_{t_2}}{k_t}\,\tilde{j} \tag{B4}$$

where

$$R = [(x_b-x_a)^2 + (y_b-y_a)^2]^{1/2} \tag{B5}$$

and k_t is the length of the antenna pair.

Hence,

$$\cos\theta = \tilde{P}_t \cdot \tilde{P}_b = \begin{pmatrix} \frac{x_b-x_a}{R} & \frac{y_b-y_a}{R} \end{pmatrix} \begin{pmatrix} \frac{x_{t_1}-x_{t_2}}{k_t} \\ \frac{y_{t_1}-y_{t_2}}{k_t} \end{pmatrix} . \tag{B6}$$

Let

$$c_1 = \frac{x_{t_1}-x_{t_2}}{k_t} \quad \text{and} \quad c_2 = \frac{y_{t_1}-y_{t_2}}{k_t} \tag{B7),(B8}$$

then

$$\cos\theta = c_1\,\frac{x_b-x_a}{R} + c_2\,\frac{y_b-y_a}{R} . \tag{B9}$$

Differentiating with respect to the state variables yields

$$\frac{\partial(\cos\theta)}{\partial x_b} = \frac{c_1}{R} - \left[\frac{c_1(x_b-x_a) + c_2(y_b-y_a)}{R^2}\right]\frac{\partial R}{\partial x_b} \tag{B10}$$

since,

$$\frac{\partial R}{\partial x_b} = \frac{x_b-x_a}{R} , \tag{B11}$$

$$\frac{\partial(\cos\theta)}{\partial x_b} = \frac{c_1}{R} - \cos\theta\,\frac{x_b-x_a}{R^2} \tag{B12}$$

similarly,

$$\frac{\partial(\cos\ \theta)}{\partial y_b} = \frac{c_2}{R} - \cos\ \theta\ \frac{y_b - y_a}{R^2}\ . \tag{B13}$$

In matrix notation,

$$\frac{\partial(\cos\ \theta)}{\partial X_b} = \frac{1}{|\vec{P}_b|}\ [\tilde{P}_t - (\tilde{P}_t \cdot \tilde{P}_b)\tilde{P}_b]\ . \tag{B14}$$

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13. ABSTRACT

In 1960 R. E. Kalman introduced a least square concept that gives optimal estimates for the state of some dynamic systems. Included is a brief historical introduction leading to his work, a summary of his work, and the application of the theory to the sonobuoy reference system used on the S3A aircraft. Also, a tutorial development of certain quantities used in the filter is presented in appendices A and B.

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